

TEMA. 3.- Transformada de Laplace

$$T(s) = \int_{-\infty}^{\infty} N(t, s) f(t) dt$$

$$\text{Laplace} \Rightarrow N(t, s) = \begin{cases} 0; t < 0 \\ e^{-st}; t \geq 0 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} [e^{-st}] f(t) dt$$

$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st} (1) dt$$

$$= \left[\int e^{-st} dt \right]_0^{\infty}$$

$$\int e^{-st} dt = -\frac{1}{s} \int -s e^{-st} dt$$

$$= -\frac{1}{s} \left[e^{-st} \right]_0^{\infty}$$

$$= -\frac{1}{s} \left(\lim_{b \rightarrow \infty} e^{-sb} - e^{-s(0)} \right)$$

$$\lim_{b \rightarrow \infty} e^{-sb} = \lim_{b \rightarrow \infty} \frac{1}{e^{sb}}$$

$$= \lim_{a \rightarrow \infty} \frac{1}{a} = 0$$

$$\mathcal{L}\{1\} = -\frac{1}{s} (0 - 1)$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}}$$

$$\begin{aligned} \mathcal{L}\{t\} &= \int_0^{\infty} e^{-st} t \, dt \\ &= \int_0^{\infty} t e^{-st} \, dt \end{aligned}$$

$$\int u \, dv = uv - \int v \, du \quad \begin{array}{l} u=t \quad du=dt \\ dv=e^{-st} \quad v=-\frac{1}{s}(e^{-st}) \end{array}$$

$$\begin{aligned} \int t e^{-st} \, dt &= t \cdot \left(-\frac{1}{s} e^{-st}\right) - \int -\frac{1}{s} (-s e^{-st}) \, dt \\ &= -\frac{t}{s} e^{-st} + \frac{1}{s} e^{-st} \end{aligned}$$

$$\begin{aligned} \mathcal{L}\{t\} &= -\frac{1}{s} \left[t e^{-st} \right]_0^{\infty} + \frac{1}{s^2} \left[e^{-st} \right]_0^{\infty} \\ &= -\frac{1}{s} \left[\lim_{b \rightarrow \infty} b \cdot e^{-sb} - (0) \cdot 1 \right] + \frac{1}{s^2} \left[\lim_{b \rightarrow \infty} e^{-sb} - 1 \right] \end{aligned}$$

$$\begin{aligned} &= \lim_{b \rightarrow \infty} b \cdot e^{-sb} = \lim_{b \rightarrow \infty} b \cdot \lim_{b \rightarrow \infty} \frac{1}{e^{sb}} \\ &= (+\infty) \cdot (0) \\ &\quad \underbrace{\hspace{1cm}}_{(0)} \end{aligned}$$

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}}$$

$$\begin{aligned}
\mathcal{L}\{e^{at}\} &= \int_0^{\infty} e^{-st} e^{at} dt \\
&= \left[\int e^{-(s-a)t} dt \right]_0^{\infty} \\
&= -\frac{1}{s-a} \left(\int (s-a) e^{-(s-a)t} dt \right)_0^{\infty} \\
&= -\frac{1}{s-a} \left(\lim_{b \rightarrow \infty} e^{-(s-a)b} - e^{(0)} \right) \\
&= -\frac{1}{s-a} (0 - 1)
\end{aligned}$$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}}$$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^n \quad n=1,2,3\dots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$t \sin(bt)$	$\frac{2sb}{(s^2+b^2)^2}$
$t \cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
$e^{at} \sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$

$$F(s) \longrightarrow f(t)$$
$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} F(s) ds$$

$$\frac{d^2 y}{dt^2} + \frac{dy}{dt} + y = e^{2t} + \cos(4t) \quad y(0)=2 \quad y'(0)=-3$$

$$\mathcal{L}\left\{\frac{d^2 y}{dt^2} + \frac{dy}{dt} + y\right\} = \mathcal{L}\{e^{2t} + \cos(4t)\}$$

$$\mathcal{L}\left\{\frac{d^2 y}{dt^2}\right\} + \mathcal{L}\left\{\frac{dy}{dt}\right\} + \mathcal{L}\{y\} = \mathcal{L}\{e^{2t}\} + \mathcal{L}\{\cos(4t)\}$$

$$(\cancel{s^2/4} - s(2) - (-3)) + (\cancel{s/4} - (2)) + \cancel{y} = \frac{1}{s-2} + \frac{s}{s^2+16}$$

$$(s^2 + s + 1)\mathcal{L}\{y\} - 2s + 1 = \frac{1}{s-2} + \frac{s}{s^2+16}$$

$$(s^2 + s + 1)\mathcal{L}\{y\} = \frac{1}{s-2} + \frac{s}{s^2+16} + 2s - 1$$

$$= \frac{(s^2+16) + s(s-2) + (2s-1)(s-2)(s^2+16)}{(s-2)(s^2+16)}$$

$$= \frac{s^2+16 + s^2-2s + (2s-1)(s^3-2s^2+16s-32)}{(s-2)(s^2+16)}$$

$$= \frac{s^2+16 + s^2-2s + 2s^4-5s^3+34s^2-80s+32}{(s-2)(s^2+16)}$$

$$= \frac{2s^4-5s^3+36s^2-82s+48}{(s-2)(s^2+16)}$$

$$\mathcal{L}\{y\} = \frac{2s^4-5s^3+36s^2-82s+48}{(s^2+s+1)(s-2)(s^2+16)}$$

$$\mathcal{L}\{y\} = \frac{As+B}{(s^2+s+1)} + \frac{C}{s-2} + \frac{Ds+E}{s^2+16}$$

$$y = \mathcal{L}^{-1}\left\{\frac{As+B}{(s^2+2(\frac{s}{2})+(\frac{1}{4})) + \frac{3}{4}} + \frac{C}{s-2} + \frac{Ds+E}{s^2+16}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{As+B}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right\} + \mathcal{L}^{-1}\left\{\frac{C}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{Ds+E}{s^2+16}\right\}$$

$$= A \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right\} + \frac{1}{\sqrt{3}} \mathcal{L}^{-1}\left\{\frac{(B-\frac{1}{2})\sqrt{3}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right\} + C e^{2t} + D e^t \cos(4t) + E e^t \sin(4t)$$

1.- lineal

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$$

$a, b \in \text{const.}$

2.- semejanza

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

$$\mathcal{L}\{\cos(5t)\} = \frac{1}{5} \left(\frac{s/5}{(\frac{s}{5})^2 + 1^2} \right) \quad \mathcal{L}\{\cos(t)\} = \frac{s}{s^2 + 1}$$

$$= \frac{1}{25} \left(\frac{s}{\frac{s^2}{25} + 1} \right) \Rightarrow \left(\frac{s}{\frac{s^2 + 25}{25}} \right)$$

$$\mathcal{L}\{\cos(5t)\} = \frac{1}{\cancel{25}} \left(\frac{\cancel{25}s}{s^2 + 25} \right) \Rightarrow \left(\frac{s}{s^2 + 25} \right)$$

3.- Derivada

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

4.- Derivada de la función TL

$$\mathcal{L}^{-1}\{F'(s)\} = -tf(t) \quad \mathcal{L}^{-1}\{sF(s)\} = f(t)$$

5.- Integral

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{F(s)}{s}$$

6.- Integral función TL

$$\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma) d\sigma\right\} = \frac{f(t)}{t}$$

7.- tardanza

$$\mathcal{L}\{f(t-\tau)\} = e^{-s\tau} F(s)$$

8.- desplazamiento

$$\mathcal{L}\{e^{\lambda t} f(t)\} = F(s-\lambda)$$

9.- producto (convolución)

$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f(t) * g(t)$$

$$f(t) * g(t) = \int_0^t f(z) \cdot g(t-z) dz$$

$$\mathcal{L}\{f(t) * g(t)\} = F(s) \cdot G(s)$$